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Inverse Sensitivity Method to Self-Adjoint Systems with Repeated Eigenvalues

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The first-order Taylor expansions of eigenpairs having a repeated latent root in cases when vibrating systems are described by multiparametric self-adjoint discrete models are derived. An inverse method based on the Taylor expansions is proposed for complementing and correcting Pesek's results (Pesek, L., "An Extension of the Inverse Sensitivity Method to Systems with Repeated Eigenvalues," ***Journal of Sound and Vibration*****, Vol. 182, No. 4, 1995, pp. 623–635). Two examples are presented to demonstrate the effectiveness of the present method.**

Nomenclature

$\boldsymbol{D}(\tau)$	= h-order coefficient matrix depending on direction τ
$\boldsymbol{e}_i$	= ith natural base vector of the n parametric space
h	= multiplicity of the eigenvalue
$\boldsymbol{I}_h$	= h-order identity matrix
$\boldsymbol{J}$	= Jacobian matrix
$\mathcal{K}$	= frequency shift stiffness matrix
$\boldsymbol{K}(\boldsymbol{p})$	= structural stiffness matrix depending on $\boldsymbol{p}$
$\boldsymbol{M}(\boldsymbol{p})$	= structural mass matrix depending on $\boldsymbol{p}$
$\boldsymbol{p}$	= vector of structural physical parameters of dimension N
$\boldsymbol{X}$	= matrix of any h M-orthogonal eigenvectors of multiple eigenvalue
$\boldsymbol{Z}$	= matrix of h M-orthogonal derivable eigenvectors of multiple eigenvalue
Λ	= diagonal matrix of h eigenvalues
λ	= h multiplicity of repeated eigenvalue
τ	= unit vector of any direction in n parametric space

Introduction

THE orthonormalized eigenvector set corresponding to a repeated eigenvalue λ (abbreviate λ base), and its derivatives with respect to design parameters, have degenerative and directive properties at a given point $P_0(\boldsymbol{p}_0)$ in the multiparametric space, respectively, i.e., there are infinite number λ bases at P_0, and there exist distinct derivable λ bases and its directional derivatives along different directions passing P_0. Moreover, as multivariable functions the λ and λ base (repeated root eigenpairs) are in general not differentiable, although their directional derivatives exist. The singularity and complexity of repeated root eigenpairs motivate us to reconsider their Jacobian matrix at point P_0.

Ojalvo and Dailey are pioneers in the research field of eigenderivatives. Dailey[1] improved Ojalvo's[2] method, and the methods of Dailey and of Juang et al.[3] were further developed by Zhang and Wang.[4] Using Dailey's ideas, Pesek[5] presented an extension of the inverse sensitivity method to conservative systems with repeated eigenvalues. However, this extension applies to some very special cases only. It caused an error in Eqs. (2.8–2.10) of his paper. Obviously, the eigenvector matrix $\boldsymbol{X}$ is discontinuous as he said in Ref. 5, and so its derivative $\boldsymbol{X}'$ does not exist in general.

A general inverse sensitivity method of self-adjoint systems is developed in the present study; not only is Pesek's method corrected, but also the first-order Taylor expansion of a repeated eigenvalue is complemented.

Several Assumptions

To explain the essence of the problem, without loss of generality the discussions are carried out under the following assumptions at some neighborhood of a design point $P = P_0$:

1) $\boldsymbol{K}(\boldsymbol{p})$ and $\boldsymbol{M}(\boldsymbol{p})$ are $N \times N$ real symmetric matrices, $\boldsymbol{M}(\boldsymbol{p})$ is positive definite, and $\boldsymbol{K}(\boldsymbol{p})$ is positive semidefinite. Consequently, the vibrating systems $[\boldsymbol{K}(\boldsymbol{p}), \boldsymbol{M}(\boldsymbol{p})]$ are definite and nondefective.

2) Various orders of continuous partial derivatives of $\boldsymbol{K}(\boldsymbol{p})$ and $\boldsymbol{M}(\boldsymbol{p})$ exist. Consequently, their directional derivatives and total differential can be found according to the following formulas[6]:

Directional derivatives:

$$\boldsymbol{F}_{,\tau} = \left.\frac{\partial \boldsymbol{F}(\boldsymbol{p})}{\partial \tau}\right|_{P=P_0} = \lim_{\substack{\Delta\tau \to 0 \\ \text{along } \tau}} \frac{\boldsymbol{F}(\boldsymbol{p}) - \boldsymbol{F}(\boldsymbol{p}_0)}{\Delta \tau} = \sum_{i=1}^{n} \boldsymbol{F}_{,i} \cos\alpha_{\tau e_i} \tag{1a}$$

$$\boldsymbol{F}_{,\tau\tau} = \sum_{i,j=1}^{n} \boldsymbol{F}_{,ij} \cos\alpha_{\tau e_i} \cos\alpha_{\tau e_j} \tag{1b}$$

where

$$\Delta\tau = |\Delta \boldsymbol{p}| = \sqrt{\sum_{i=1}^{n} (\Delta p_i)^2} \tag{2}$$

$$\boldsymbol{F}_{,i} = \left.\frac{\partial \boldsymbol{F}(\boldsymbol{p})}{\partial p_i}\right|_{P=P_0}, \qquad \boldsymbol{F}_{,ij} = \left.\frac{\partial^2 \boldsymbol{F}(\boldsymbol{p})}{\partial p_i \partial p_j}\right|_{P=P_0} \qquad i, j = 1, 2, \ldots, n \tag{3}$$

and $\cos\alpha_{\tau e_i}$, $i = 1, 2, \ldots, n$ directional cosines of τ, respectively.

Total differential:

$$\mathrm{d}\boldsymbol{F} = \sum_{i=1}^{n} \boldsymbol{F}_{,i}\, \mathrm{d}p_i \tag{4}$$

where $\boldsymbol{F}_{,i}$ depends only on the P_0, whereas it is independent of Δp_j, $j = 1, 2, \ldots, n$

3)

$$\Lambda(\boldsymbol{p}) = \mathrm{diag}\left[\lambda^{(1)}(\boldsymbol{p}), \lambda^{(2)}(\boldsymbol{p}), \ldots, \lambda^{(h)}(\boldsymbol{p})\right]$$

$$0 \le \lambda^{(1)}(\boldsymbol{p}) < \lambda^{(2)}(\boldsymbol{p}) < \cdots < \lambda^{(h)}(\boldsymbol{p})$$

$$\boldsymbol{p} \ne \boldsymbol{p}_0, \qquad 1 \le h \le N \tag{5a}$$

$$\boldsymbol{Z}(\boldsymbol{p}) = \left[z^{(1)}(\boldsymbol{p}), z^{(2)}(\boldsymbol{p}), \ldots, z^{(h)}(\boldsymbol{p})\right] \tag{5b}$$

are eigenpairs of $[\boldsymbol{K}(\boldsymbol{p}), \boldsymbol{M}(\boldsymbol{p})]$ i.e., $[\Lambda(\boldsymbol{p}), \boldsymbol{Z}(\boldsymbol{p})]$ are solutions of the generalized eigenproblem

$$\boldsymbol{K}(\boldsymbol{p})\boldsymbol{Z}(\boldsymbol{p}) = \boldsymbol{M}(\boldsymbol{p})\boldsymbol{Z}(\boldsymbol{p})\Lambda(\boldsymbol{p}) \tag{6a}$$

$$\boldsymbol{Z}^T(\boldsymbol{p})\boldsymbol{M}(\boldsymbol{p})\boldsymbol{Z}(\boldsymbol{p}) = \boldsymbol{I}_h \tag{6b}$$

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and have the following limit properties:

$$\lim_{P \to P_0} \Lambda(\boldsymbol{p}) = \lambda \boldsymbol{I}_h, \qquad \lim_{\substack{P \to P_0 \\ \text{along } \tau}} \boldsymbol{Z}(\boldsymbol{p}) = \boldsymbol{Z}(\tau) \tag{7}$$

$$\lim_{P \to P_0} \lambda^{(s)}(\boldsymbol{p}) \neq \lambda, \qquad s \neq 1, 2, \ldots, h \tag{8}$$

Differentiability of the Repeated Root Eigenpairs

The directional derivatives of the repeated root eigenpairs can be obtained by means of single parameter formulas; this parameter is just the length $\Delta\tau = |\Delta \boldsymbol{p}| = \overline{P_0 P}$ along direction τ passing P_0. $\Lambda_{,\tau}$ and $\boldsymbol{Z}(\tau)$ can be determined by the following standard eigenproblem[7,8]:

$$\boldsymbol{A}(\tau, \boldsymbol{X})\Gamma(\tau, \boldsymbol{X}) = \Gamma(\tau, \boldsymbol{X})\Lambda_{,\tau} \tag{9a}$$

$$\Gamma^T(\tau, \boldsymbol{X})\Gamma(\tau, \boldsymbol{X}) = \boldsymbol{I}_h \tag{9b}$$

where

$$\boldsymbol{A}(\tau, \boldsymbol{X}) = \boldsymbol{X}^T \mathcal{K}_\tau \boldsymbol{X} \tag{10}$$

and $\boldsymbol{Z}(\tau)$ is given by

$$\boldsymbol{Z}(\tau) = \boldsymbol{X}\Gamma(\tau, \boldsymbol{X}) \tag{11}$$

We denote

$$\mathcal{K} = \boldsymbol{K} - \lambda \boldsymbol{M} \tag{12}$$

and notations

$$\mathcal{K}_\tau = \boldsymbol{K}_{,\tau} - \lambda \boldsymbol{M}_{,\tau} \tag{13a}$$

$$\mathcal{K}_{\tau\tau} = \boldsymbol{K}_{,\tau\tau} - \lambda \boldsymbol{M}_{,\tau\tau} \tag{13b}$$

are introduced to simplify the writing process.

Note that $\boldsymbol{Z}(\tau)$ depends only on the direction τ, whereas it is independent of the chosen λ base, say $\boldsymbol{X}$. Here we further suppose that the diagonal matrix $\Lambda_{,\tau}$ has distinct diagonal entries $\lambda_\tau^{(1)} < \lambda_\tau^{(2)} < \cdots < \lambda_\tau^{(h)}$, then $\Gamma(\tau, \boldsymbol{X})$ is determined uniquely by Eq. (9). From Eqs. (9–11), we may get

$$\Lambda_{,\tau} = \boldsymbol{Z}^T(\tau)\mathcal{K}_\tau \boldsymbol{Z}(\tau) \tag{14}$$

in Eqs. (14) and (11) taking $\tau = \boldsymbol{e}_i$, $i = 1, 2, \ldots, n$, respectively, the partial derivatives of λ with respect to p_i and its derivable λ bases can be obtained as

$$\Lambda_{,i} = \boldsymbol{Z}^T(\boldsymbol{e}_i)\mathcal{K}_{e_i}\boldsymbol{Z}(\boldsymbol{e}_i) \tag{15a}$$

$$\boldsymbol{Z}(\boldsymbol{e}_i) = \boldsymbol{X}\Gamma(\boldsymbol{e}_i, \boldsymbol{X}), \qquad i = 1, 2, \ldots, n \tag{15b}$$

from which the transforms of derivable λ bases along various directions can be derived to be

$$\boldsymbol{Z}(\boldsymbol{e}_i) = \boldsymbol{Z}(\tau)\Theta(\tau, \boldsymbol{e}_i) \tag{16a}$$

$$\Theta^T(\tau, \boldsymbol{e}_i)\Theta(\tau, \boldsymbol{e}_i) = \boldsymbol{I}_h \tag{16b}$$

where

$$\Theta(\tau, \boldsymbol{e}_i) = \Gamma^T(\tau, \boldsymbol{X})\Gamma(\boldsymbol{e}_i, \boldsymbol{X}) \tag{17}$$

Obviously, the orthogonal matrices $\Theta(\tau, \boldsymbol{e}_i)$, $i = 1, 2, \ldots, n$ are independent of $\boldsymbol{X}$. Using Eqs. (13–16) and (1), we get the directional derivative formulae of a repeated eigenvalue at P_0 along any direction τ:

$$\Lambda_{,\tau} = \sum_{i=1}^{n} \Theta(\tau, \boldsymbol{e}_i)\left[\boldsymbol{Z}^T(\boldsymbol{e}_i)\mathcal{K}_{e_i}\boldsymbol{Z}(\boldsymbol{e}_i)\right]\Theta^T(\tau, \boldsymbol{e}_i)\cos\alpha_{\tau e_i}$$
$$= \sum_{i=1}^{n}\left[\Theta(\tau, \boldsymbol{e}_i)\Lambda_{,i}\Theta^T(\tau, \boldsymbol{e}_i)\right]\cos\alpha_{\tau e_i} \tag{18}$$

Making a comparison between Eqs. (1a) and (18) allows us to conclude that the repeated eigenvalue λ is in general not differentiable because generally $\Theta(\tau, \boldsymbol{e}_i) \fallingdotseq \boldsymbol{I}_h$, $\forall \tau \in R^n$, and we can also conclude that as the functions of design parameters the partial derivatives $\Lambda_{,i}$ $i = 1, 2, \ldots, n$ of a repeated eigenvalue are definitely not all continuous.

From Eq. (18) we get

$$\lambda_{,\tau}^{(q)}\delta_{qr} = \sum_{i=1}^{n}\sum_{s=1}^{h}\lambda_{,i}^{(s)}\left[\theta^{qs}(\tau, \boldsymbol{e}_i)\theta^{rs}(\tau, \boldsymbol{e}_i)\right]\cos\alpha_{\tau e_i}$$
$$q, r = 1, 2, \ldots, h \tag{19}$$

The directional derivatives of the derivable λ base $\boldsymbol{Z}(\tau)$ at P_0 along any direction τ can be decomposed uniquely as

$$\boldsymbol{Z}_{,\tau} = \hat{\boldsymbol{Z}}_{,\tau} + \tilde{\boldsymbol{Z}}_{,\tau} \tag{20}$$

where

$$\hat{\boldsymbol{Z}}_{,\tau} = \boldsymbol{Z}(\tau)\boldsymbol{D}(\tau) \tag{21a}$$

$$\tilde{\boldsymbol{Z}}_{,\tau} = -\boldsymbol{G}[\mathcal{K}_\tau \boldsymbol{Z}(\tau)] \tag{21b}$$

are simply called *nuclear* components of $\mathcal{K}$ and its $\boldsymbol{M}$-orthogonal complement component, respectively. In the preceding second formula $\boldsymbol{G}$ is a constraint-generalized inverse of $\mathcal{K}$:

$$\boldsymbol{G} = \boldsymbol{P}^T \mathcal{K}^{\{1\}}\boldsymbol{P} \tag{22}$$

where

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{X}\boldsymbol{X}^T \tag{23a}$$

$$\forall \mathcal{K}^{\{1\}} \in \mathcal{K}\{1\} : (\boldsymbol{Q} \mid \mathcal{K}\boldsymbol{Q}\mathcal{K} = \mathcal{K}) \tag{23b}$$

are idempotent matrix and generalized {1} inverse[9] of $\mathcal{K}$, respectively, and the entries of $h \times h$ matrix $\boldsymbol{D}(\tau)$ can be given by Ref. 4:

$$d^{qq}(\tau) = 0.5 l^{qq}(\tau), \qquad q = 1, 2, \ldots, n \tag{24}$$

$$d^{qr}(\tau) = \frac{u^{qr}(\tau)}{\lambda_{,\tau}^{(r)} - \lambda_{,\tau}^{(q)}} q \neq r, \qquad q, r = 1, 2, \ldots, n \tag{25}$$

where $d^{qr}(\tau)$, $l^{qr}(\tau)$, and $u^{qr}(\tau)$ are qth row and rth column entries in $h \times h$ matrices $\boldsymbol{D}(\tau)$, $\boldsymbol{L}(\tau)$, and $\boldsymbol{U}(\tau)$, respectively:

$$\boldsymbol{L}(\tau) = -\boldsymbol{Z}^T(\tau)\boldsymbol{M}_{,\tau}\boldsymbol{Z}(\tau) = \sum_{i=1}^{n}\Theta(\tau, \boldsymbol{e}_i)\boldsymbol{L}(\boldsymbol{e}_i)\Theta^T(\tau, \boldsymbol{e}_i)\cos\alpha_{\tau e_i} \tag{26}$$

$$\boldsymbol{U}(\tau) = \boldsymbol{Z}^T(\tau)(0.5\mathcal{K}_{\tau\tau} - \mathcal{K}_\tau \boldsymbol{G}\mathcal{K}_\tau)\boldsymbol{Z}(\tau) + \boldsymbol{L}(\tau)\Lambda_{,\tau}$$
$$= \sum_{i=1}^{n}\Theta(\tau, \boldsymbol{e}_i)\boldsymbol{U}(\boldsymbol{e}_i)\Theta^T(\tau, \boldsymbol{e}_i)\cos^2\alpha_{\tau e_i}$$
$$+ \sum_{i \neq j}\Theta(\tau, \boldsymbol{e}_i)\big[\boldsymbol{L}(\boldsymbol{e}_i)\Theta^T(\tau, \boldsymbol{e}_i)\Theta(\tau, \boldsymbol{e}_j)\Lambda_{,j}$$
$$+ \boldsymbol{Z}^T(\boldsymbol{e}_i)\big(0.5\mathcal{K}_{e_i e_j} - \mathcal{K}_{e_i}\boldsymbol{G}\mathcal{K}_{e_j}\big)\boldsymbol{Z}(\boldsymbol{e}_j)\big]\Theta^T(\tau, \boldsymbol{e}_j)$$
$$\times \cos\alpha_{\tau e_i}\cos\alpha_{\tau e_j} \tag{27}$$

Using Eqs. (13a), (16a), and (1a), from Eq. (21b) we can get

$$\tilde{\boldsymbol{Z}}_{,\tau} = -\sum_{i=1}^{n}\boldsymbol{G}\mathcal{K}_{e_i}\boldsymbol{Z}(\boldsymbol{e}_i)\Theta^T(\tau, \boldsymbol{e}_i)\cos\alpha_{\tau e_i}$$
$$= \sum_{i=1}^{n}\left[\tilde{Z}_{,i}\Theta^T(\tau, \boldsymbol{e}_i)\right]\cos\alpha_{\tau e_i} \tag{28}$$

or equivalently

$$\tilde{z}_{,\tau}^{(s)} = \sum_{i=1}^{n}\left[\sum_{r=1}^{h}\tilde{z}_{,i}^{(r)}\theta^{sr}(\tau, \boldsymbol{e}_i)\right]\cos\alpha_{\tau e_i}, \qquad s = 1, 2, \ldots, h \tag{29}$$

Now, let us suppose Pesek's result

$$Z_{,\tau} = \sum_{i=1}^{n} Z_{,i}\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i} \tag{30}$$

holds; then we can get the following equation easily:

$$\hat{Z}_{,\tau} = \sum_{i=1}^{n} \hat{Z}_{,i}\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i} \tag{31}$$

Using Eqs. (11), (15b), (21a), and (31), we have

$$X\Gamma(\tau, X)D(\tau) = X\sum_{i=1}^{n}\Gamma(e_i, X)D(e_i)\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i} \tag{32}$$

Notice that X is a λ base that has full column rank and that $\Gamma(\tau, X)$ is an orthogonal matrix. From Eq. (32) we can immediately derive as follows:

$$D(\tau) = \sum_{i=1}^{n}\Gamma^T(\tau, X)\Gamma(e_i, X)D(e_i)\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i}$$

$$= \sum_{i=1}^{n}\Theta(\tau, e_i)D(e_i)\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i} \tag{33}$$

Conversely, if equality (33) holds, we also can derive

$$\hat{Z}_{,\tau} = \sum_{i=1}^{n} Z(\tau)\Theta(\tau, e_i)D(e_i)\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i}$$

$$= \sum_{i=1}^{n} Z(e_i)\Theta^T(\tau, e_i)\Theta(\tau, e_i)D(e_i)\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i}$$

$$= \sum_{i=1}^{n} \hat{Z}_{,i}\Theta^T(\tau, e_i)\cos\alpha_{\tau e_i}$$

Then Eq. (30) will hold.

Pesek's result (30) will hold if and only if Eq. (33) holds. However, equality (33) does not hold in general except for a few special cases. Therefore, we can say that Pesek's inverse sensitivity method is not suitable for most cases.

It is important to realize, via observations of Eqs. (24–27), that the nuclear component $\hat{Z}_{,\tau}$ depends heavily on the direction τ, and its computations would involve the second partial derivatives of stiffness and mass matrices. Although the derivable λ base $Z(\tau)$ and its directional derivative $Z_{,\tau}$ exist, yet $Z(\tau)$ is in general not differentiable with respect to design as well as the repeated eigenvalue. Finally we point out that Pesek's result (30) is applicable to the following case:

$$D^T(\tau) = D(\tau), \qquad \forall\tau \in R^n \tag{34}$$

Using Eq. (26) and the following equation[7]

$$D^T(\tau) + D(\tau) = L(\tau) \tag{35}$$

we can easily prove the truthfulness of the preceding conclusion.

Inverse Sensitivity Method for the Cases of Repeated Eigenvalues

In the problem for the parametric corrections of mechanical systems, the analytical models $[K(p), M(p)]$ are generally given. Suppose that the eigenpairs of an h multiple eigenvalue at P_0 has been computed as

$$\lambda I_h, \qquad X = \lfloor x^{(1)} \quad x^{(2)} \quad \cdots \quad x^{(h)} \rfloor \tag{36}$$

By contrast with the corresponding test datum

$$\bar{\Lambda} = \mathrm{diag}\big(\bar{\lambda}^{(1)} \quad \bar{\lambda}^{(2)} \quad \cdots \quad \bar{\lambda}^{(h)}\big)$$
$$\bar{Z} = \lfloor \bar{z}^{(1)} \quad \bar{z}^{(2)} \quad \cdots \quad \bar{z}^{(h)} \rfloor \tag{37}$$

there exist the differences. Try using the parametric corrections

$$p = p_0 + \Delta p \tag{38}$$

such that the system $[K(p), M(p)]$ has eigenpairs, which have evolved from the $(\lambda I_h, X\Gamma)$, consistent with test datum $(\bar{\Lambda}, \bar{Z})$. This problem is to take the gauge of the correction Δp satisfying the preceding requirements. To this end we take into account only the first-order Taylor expansions of eigenpairs by Eq. (5) along need-determining direction τ as [see Eq. (19)]:

$$\bar{\lambda}^{(q)} - \lambda = \lambda_{,\tau}^{(q)}\cdot\Delta\tau = \sum_{i=1}^{n}\left\{\sum_{s=1}^{h}\lambda_{,i}^{(s)}\big[\theta^{qs}(\tau, e_i)\big]^2\right\}\Delta p_i$$
$$q = 1, 2, \ldots, h \tag{39}$$

$$\bar{Z} - Z(\tau) = Z_{,\tau}\cdot\Delta\tau = \tilde{Z}_{,\tau}\cdot\Delta\tau + \hat{Z}_{,\tau}\cdot\Delta\tau \tag{40}$$

Owing to so much trouble for computing nuclear component $\hat{Z}_{,\tau}$, we substitute Eq. (40) with another equation. Premultiplying Eq. (40) by the projection operator P^T [see Eq. (23a)], we get

$$P^T\bar{Z} = \tilde{Z}_{,\tau}\cdot\Delta\tau = \sum_{i=1}^{n}\big[\tilde{Z}_{,i}\Theta^T(\tau, e_i)\big]\Delta p_i \tag{41}$$

or [see Eq. (29)]

$$P^T\bar{z}^{(s)} = \sum_{i=1}^{n}\left[\sum_{r=1}^{h}\tilde{z}_{,i}^{(r)}\theta^{sr}(\tau, e_i)\right]\Delta p_i \tag{42}$$

Equations (42) and (39) can be expressed as a matrix equation

$$J\Delta p = \Delta v \tag{43}$$

where

$$\Delta p = \underset{i}{\mathrm{col}}(\Delta p_i), \qquad i = 1, 2, \ldots, n \tag{44}$$

$$\Delta v = \left\{\frac{\underset{s}{\mathrm{col}}\big[P^T\bar{z}^{(s)}\big]}{\underset{q}{\mathrm{col}}[\bar{\lambda}^{(q)} - \lambda]}\right\}, \qquad J = \left[\frac{\underset{s}{\mathrm{col}}(w_s)}{\underset{q}{\mathrm{col}}(t_q)}\right]$$
$$q, s = 1, 2, \ldots, h \tag{45}$$

$$w_s = \underset{i}{\mathrm{row}}\left[\sum_{r=1}^{h}\tilde{z}_{,i}^{(r)}\theta^{sr}(\tau, e_i)\right]$$

$$t_q = \underset{i}{\mathrm{row}}\left\{\sum_{s=1}^{h}\lambda_{,i}^{(s)}\big[\theta^{qs}(\tau, e_i)\big]^2\right\}$$

The Jacobian matrix J involves entries of $\Theta(\tau, e_i)$, $i = 1, 2, \ldots, n$, which depend on Δp to be determined, so that it is difficult to solve Eq. (43) directly. To solve Eq. (43) linearly, the continuity of $Z(\tau)$, which is expressed by Eq. (8), is employed, that is, $Z(\tau) = X\Gamma(\tau, X)$ would lie adjacent enough to $\bar{Z}$ as $\Delta\tau$ is sufficiently small. In view of the preceding, an estimation of $\Gamma(X, \tau)$ is cited by Ref. 3 as follows:

$$\Gamma(\tau, X) = (X^T\bar{Z}\bar{Z}^TX)^{\frac{1}{2}}(\bar{Z}^TX)^{-1} \tag{46}$$

which minimizes the trace of the matrix $(X\Gamma - \bar{Z})^T(X\Gamma - \bar{Z})$, in the sense of least squares under the constraint $\Gamma^T\Gamma = I$. Substituting the preceding $\Gamma(\tau, X)$ into Eq. (17), we obtain the required matrices $\Theta(\tau, e_i)$, $i = 1, 2, \ldots, n$. Generally, Eq. (43) is overdetermined. This system can be solved, for example, in the sense of least squares. Then an unknown vector Δp is expressed as

$$\Delta p = [J^TJ]^{-1}J^T\Delta v = J^+\Delta v \tag{47}$$

where J^+ is the Moore–Penrose generalized inverse of matrix J. After computing Δp, a new estimation of parametric vector p can be determined by Eq. (38). The new $[K(p), M(p)]$ are then assembled, and the calculation of Δp is repeated. This iteration process leads to the minimization of the model-structure modal and spectral difference.

Two Examples

The first example is chosen from Eq. (4.1) of Ref. 5. In this very special case not only the repeated eigenvalue λ is differentiable, but the derivable λ base is also differentiable. Therefore, this example cited by Pesek has not been able to demonstrate the necessity of his extension. The latter example shows that Eq. (2.11) in Ref. 5 is generally not applicable, and we must use the Jacobian matrix (45) by the authors to get the correct results.

Stiffness and Mass Matrices

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \qquad M = \begin{bmatrix} m_1 + m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix}$$

This vibrating system has a repeated eigenvalue $\lambda = k/m$ when the stiffness $k_1 = k_2 = k$ and masses $m_1 = m_2 = m$. This system with equal mass and stiffness is called *reference system* by Pesek. The corresponding orthonormal eigenvectors are degenerate, and then we can choose arbitrarily the following initial λ base:

$$X = \begin{bmatrix} x^{(1)} & x^{(2)} \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Now let m_i, $i = 1, 2$ be the design parameters and let $k_i \equiv 1$, $m_{i0} = 10$, $i = 1, 2$, we then have

$$\lambda = 0.1, \qquad X = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$M_{,1} = \left. \frac{\partial M}{\partial m_1} \right|_{P = P_0} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_{,2} = \left. \frac{\partial M}{\partial m_2} \right|_{P = P_0} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The partial derivatives $K_{,i}$, $i = 1, 2$ and the higher derivatives of M and K are obviously equal to zeros. Moreover owing to the multiplicity of λ equals the order of the system, so that the frequency shift stiffness matrix $\mathcal{K}$ and its constraint generalized inverse G are also equal to zeros, respectively, that is,

$$\mathcal{K} = 0, \qquad G = 0$$

Hence, the directional derivative $Z_{,\tau}$ has only a nuclear component [see Eq. (21)]:

$$Z_{,\tau} = Z(\tau) D(\tau)$$

Using Eq. (1), from Eq. (13) we can obtain

$$\mathcal{K}_\tau = -\lambda M_{,\tau} = -\lambda(M_{,1} \cos\alpha + M_{,2} \sin\alpha)$$

$$= -0.1 \begin{bmatrix} \cos\alpha + \sin\alpha & -\sin\alpha \\ -\sin\alpha & \sin\alpha \end{bmatrix}$$

$$\mathcal{K}_{\tau\tau} = 0$$

Thus, the matrix $A(\tau, X)$ can be computed as

$$A(\tau, x) = X^T \mathcal{K}_\tau X$$

$$= -5 \times 10^{-3} \begin{bmatrix} \cos\alpha + \sin\alpha & \cos\alpha - \sin\alpha \\ \cos\alpha - \sin\alpha & \cos\alpha + \sin\alpha \end{bmatrix}$$

The solutions of the eigenproblem of $A(\tau, X)$ are found to be

$$\Lambda_{,\tau} = -10^{-2} \mathrm{diag}(\cos\alpha, \sin\alpha)$$

$$\Gamma(\tau, X) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note that the preceding orthogonal matrix $\Gamma(\tau, X)$ is independent of $\tau(\alpha)$, thus the derivable λ base $Z(\tau)$ along any direction τ at P_0

$$Z = X\Gamma(\tau, X) = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

is definitely independent of τ; it follows that [see Eq. (17)]

$$\Theta(\tau, e_i) = I_2 \qquad \forall\alpha \in [0, 2\pi)$$

Let $\tau = e_i$, $i = 1, 2$, that is, taking $\alpha = 0$ and $\pi/2$ in $\Lambda_{,\tau}$, respectively, we get

$$\Lambda_{,1} = -10^{-2} \mathrm{diag}(1, 0) \Lambda_{,2} = -10^{-2} \mathrm{diag}(0, 1)$$

we then have

$$\Lambda_{,\tau} = \Lambda_{,1} \cos\alpha + \Lambda_{,2} \sin\alpha$$

This equality shows that the repeated eigenvalue is differentiable at P_0 in this special example. The diagonal entries in $D(\tau)$ equal half of the corresponding entries of the matrix [see Eq. (24)]:

$$L(\tau) = -Z^T M_{,\tau} Z = -0.1 \,\mathrm{diag}(\cos\alpha, \sin\alpha)$$

so that

$$d^{qq}(\tau) = \begin{cases} -0.05 \cos\alpha, & q = 1 \\ -0.05 \sin\alpha, & q = 2 \end{cases} \qquad \forall\alpha \in [0, 2\pi)$$

To compute the off-diagonal entries of $D(\tau)$, we must precompute the matrix [see Eq. (27)]:

$$U(\tau) = L(\tau) \Lambda_{,\tau} = 10^{-3} \mathrm{diag}(\cos^2\alpha, \sin^2\alpha)$$

According to Eq. (25), we get

$$d^{qr}(\tau) = 0, \qquad q \neq r, \qquad q, r = 1, 2, \qquad \forall\alpha \in [0, 2\pi)$$

and it follows that

$$D(\tau) = -0.05 \,\mathrm{diag}(\cos\alpha, \sin\alpha) = D^T(\tau), \qquad \forall\alpha \in [0, 2\pi)$$

$$Z_{,\tau} = ZD(\tau) = Z[D(e_1) \cos\alpha + D(e_2) \sin\alpha]$$

$$= Z_{,1} \cos\alpha + Z_{,2} \sin\alpha$$

The preceding equality shows that the derivable λ base Z is also differentiable. Clearly, in this special example when the derivable eigenpairs with a repeat eigenvalue are all differentiable, the Pesek extension failed of necessity because their Taylor expansions of first order are the same simple eigenvalues.

Stiffness, Mass Matrices, and Design Parameters

$$K(p) = \begin{bmatrix} 1 + (a/2)q^2 & -bq \\ -bq & (a/2)q^2 \end{bmatrix}, \qquad M(p) = \begin{bmatrix} p + 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$p^T = [p, q]$$

Here, $a = 2$, $b = 1$ are given constants, and $p_0 = 1$, $q_0 = 1$ are original (reference) values. It is easy to check that the reference system

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

has a repeated eigenvalue $\lambda = 1$. There are infinite number λ bases at P_0 since the degeneracy of λ, and so we can choose one of them arbitrarily, e.g.,

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

The partial derivatives of $K(p)$ and $M(p)$ with respect to design p at P_0 are as follows:

$$K_{,1} = \left. \frac{\partial K(p)}{\partial p} \right|_{P = P_0} = 0, \qquad M_{,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$K_{,11} = \left. \frac{\partial^2 K(p)}{\partial p^2} \right|_{P = P_0} = 0, \qquad M_{,11} = 0$$

$$\boldsymbol{K}_{,2} = \frac{\partial \boldsymbol{K}(\boldsymbol{p})}{\partial q}\bigg|_{P=P_0} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{M}_{,2} = 0$$

$$\boldsymbol{K}_{,22} = 2\boldsymbol{I}_2, \qquad \boldsymbol{M}_{,22} = 0$$

$$\boldsymbol{K}_{,12} = \frac{\partial^2 \boldsymbol{K}(\boldsymbol{p})}{\partial p \partial q}\bigg|_{P=P_0} = \boldsymbol{K}_{,21} = 0$$

$$\boldsymbol{M}_{,12} = \boldsymbol{M}_{,21} = 0$$

Accordingly, we have

$$\mathcal{K}_{e_1} = -\lambda \boldsymbol{M}_{,1} = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{K}_{e_1 e_1} = 0$$

$$\mathcal{K}_{e_2} = \boldsymbol{K}_{,2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \mathcal{K}_{e_2 e_2} = \boldsymbol{K}_{,22} = 2\boldsymbol{I}_2$$

$$\mathcal{K}_{e_1 e_2} = \boldsymbol{K}_{,12} - \lambda \boldsymbol{M}_{,12} = \mathcal{K}_{e_2 e_1} = 0$$

Moreover, in view of the same reason as the first example, we have

$$\mathcal{K} = 0, \qquad \boldsymbol{G} = 0, \qquad \boldsymbol{Z}_{,\tau} = \boldsymbol{Z}(\tau)\boldsymbol{D}(\tau)$$

Thus, the directional derivatives of stiffness and mass matrices can be computed by Eq. (1), and we then can form the following matrices:

$$\mathcal{K}_\tau = \mathcal{K}_{e_1}\cos\alpha + \mathcal{K}_{e_2}\sin\alpha = \begin{bmatrix} 2\sin\alpha - \cos\alpha & -\sin\alpha \\ -\sin\alpha & 2\sin\alpha \end{bmatrix}$$

$$\mathcal{K}_{\tau\tau} = \mathcal{K}_{e_1 e_1}\cos^2\alpha + 2\mathcal{K}_{e_1 e_2}\sin\alpha\cos\alpha + \mathcal{K}_{e_2 e_2}\sin^2\alpha$$

$$= 2\sin^2\alpha\, \boldsymbol{I}_2$$

$$\boldsymbol{A}(\tau, \boldsymbol{X}) = \boldsymbol{X}^T \mathcal{K}_\tau \boldsymbol{X} = 0.5\begin{bmatrix} 2\sin\alpha - \cos\alpha & -\cos\alpha \\ -\cos\alpha & 6\sin\alpha - \cos\alpha \end{bmatrix}$$

The derivable λ base $\boldsymbol{Z}(\tau) = \boldsymbol{X}\Gamma(\tau, \boldsymbol{X})$ along any τ at P_0 and the directional derivative $\Lambda_{,\tau}$ of λ would be determined by eigenvalue problem of $\boldsymbol{A}(\tau, \boldsymbol{X})$ as

$$\Lambda_{,\tau} = \mathrm{diag}\left(\underset{\lambda_{,\tau}^{(1)}}{\underset{\Uparrow}{\frac{s(\alpha) - \tilde{h}(\alpha)}{2}}}, \underset{\lambda_{,\tau}^{(2)}}{\underset{\Uparrow}{\frac{s(\alpha) + h(\alpha)}{2}}}\right)$$

$$\Gamma(\tau, \boldsymbol{X}) = g(\alpha)\begin{bmatrix} h(\alpha) & -\cos\alpha \\ \cos\alpha & h(\alpha) \end{bmatrix}$$

where the following notations are introduced to simplify the writing process:

$$f(\alpha) = \sqrt{1 + 3\sin^2\alpha}, \qquad s(\alpha) = 2\sin\alpha - \cos\alpha$$

$$h(\alpha) = f(\alpha) + 2\sin\alpha, \qquad \tilde{h}(\alpha) = f(\alpha) - 2\sin\alpha$$

$$g(\alpha) = \frac{1}{\sqrt{2f(\alpha)h(\alpha)}} = \frac{1}{\sqrt{\cos^2\alpha + h^2(\alpha)}}$$

We now can express the derivable λ base $\boldsymbol{Z}(\tau)$ as

$$\boldsymbol{Z}(\tau) = \boldsymbol{X}\Gamma(\tau, \boldsymbol{X}) = \frac{g(\alpha)}{\sqrt{2}}\begin{bmatrix} h(\alpha) + \cos\alpha & h(\alpha) - \cos\alpha \\ 2\cos\alpha & 2h(\alpha) \end{bmatrix}$$

To compute $\boldsymbol{Z}_{,\tau}$, we have to determine $\boldsymbol{D}(\tau)$ whose diagonal and off-diagonal entries are depending on $\boldsymbol{L}(\tau)$ and $\boldsymbol{U}(\tau)$, respectively:

$$\boldsymbol{L}(\tau) = -\boldsymbol{Z}(\tau)\boldsymbol{M}_{,\tau}\boldsymbol{Z}(\tau)$$

$$= -\frac{\cos\alpha \cdot g^2(\alpha)}{2}\begin{Bmatrix} [h(\alpha) + \cos\alpha]^2 & h^2(\alpha) - \cos^2\alpha \\ h^2(\alpha) - \cos^2\alpha & [h(\alpha) - \cos\alpha]^2 \end{Bmatrix}$$

$$\boldsymbol{U}(\tau) = 0.5\boldsymbol{Z}^T(\tau)\mathcal{K}_{\tau\tau}\boldsymbol{Z}(\tau) + \boldsymbol{L}(\tau)\Lambda_{,\tau} = \frac{g^2(\alpha)}{2}(\sin^2\alpha\boldsymbol{C} - \cos\alpha\boldsymbol{B})$$

where

$$\boldsymbol{C} = \begin{bmatrix} h^2(\alpha) + 2h(\alpha)\cos\alpha + 5\cos^2\alpha & h^2(\alpha) + 4h(\alpha)\cos\alpha - \cos^2\alpha \\ h^2(\alpha) + 4h(\alpha)\cos\alpha - \cos^2\alpha & 5h^2(\alpha) - 2h(\alpha)\cos\alpha + \cos^2\alpha \end{bmatrix}$$

$$\boldsymbol{B} = \begin{Bmatrix} \lambda_{,\tau}^{(1)}[h(\alpha) + \cos\alpha]^2 & \lambda_{,\tau}^{(2)}[h^2(\alpha) - \cos^2\alpha] \\ \lambda_{,\tau}^{(1)}[h^2(\alpha) - \cos^2\alpha] & \lambda_{,\tau}^{(2)}[h(\alpha) - \cos\alpha]^2 \end{Bmatrix}$$

Using Eqs. (24) and (25), from the preceding matrices we get

$$d_{11}(\tau) = -\tfrac{1}{4}\cos\alpha \cdot g^2(\alpha)[h(\alpha) + \cos\alpha]^2$$

$$d_{22}(\tau) = -\tfrac{1}{4}\cos\alpha \cdot g^2(\alpha)[h(\alpha) - \cos\alpha]^2$$

$$d_{12}(\tau) = \frac{u_{12}(\tau)}{\lambda_{,\tau}^{(2)} - \lambda_{,\tau}^{(1)}} = \frac{g^2(\alpha)}{2f(\alpha)}\big\{\sin^2\alpha[h^2(\alpha) + 4h(\alpha)\cos\alpha - \cos^2\alpha] - \cos\alpha \cdot \lambda_{,\tau}^{(2)}[h^2(\alpha) - \cos^2\alpha]\big\}$$

$$d_{21}(\tau) = \frac{u_{21}(\tau)}{\lambda_{,\tau}^{(1)} - \lambda_{,\tau}^{(2)}} = -\frac{g^2(\alpha)}{2f(\alpha)}\big\{\sin^2\alpha[h^2(\alpha) + 4h(\alpha)\cos\alpha - \cos^2\alpha] - \cos\alpha \cdot \lambda_{,\tau}^{(1)}[h^2(\alpha) - \cos^2\alpha]\big\}$$

Substituting $\boldsymbol{D}(\tau)$ into the expression of $\boldsymbol{Z}_{,\tau} = \boldsymbol{Z}(\tau)\boldsymbol{D}(\tau)$, we obtain

$$\boldsymbol{Z}_{,\tau} = \frac{g(\alpha)}{\sqrt{2}}\begin{Bmatrix} d_{11}(\tau)[h(\alpha) + \cos\alpha] + d_{21}(\tau)[h(\alpha) - \cos\alpha] & d_{12}(\tau)[h(\alpha) + \cos\alpha] + d_{22}(\tau)[h(\alpha) - \cos\alpha] \\ 2[d_{11}(\tau)\cos\alpha + d_{21}(\tau)h(\tau)] & 2[d_{12}(\tau)\cos\alpha + d_{22}(\tau)h(\tau)] \end{Bmatrix}$$

Taking $\tau = \boldsymbol{e}_i$, $i = 1, 2$ in the preceding expressions, the derivable λ bases and its partial derivatives of λ can be derived as follows:

$$\Lambda_{,1} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Gamma(\boldsymbol{e}_1, \boldsymbol{X}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{Z}(\boldsymbol{e}_1) = \boldsymbol{X}\Gamma(\boldsymbol{e}_1, \boldsymbol{X}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Lambda_{,2} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \Gamma(\boldsymbol{e}_2, \boldsymbol{X}) = \boldsymbol{I}_2$$

$$\boldsymbol{Z}(\boldsymbol{e}_2) = \boldsymbol{X} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\boldsymbol{Z}_{,1} = -\begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \end{bmatrix}, \qquad \boldsymbol{Z}_{,2} = \frac{1}{4\sqrt{2}}\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$$

Moreover,

$$\boldsymbol{D}(\boldsymbol{e}_1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{D}(\boldsymbol{e}_2) = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Thus, the orthogonal matrices

$$\Theta(\tau, \boldsymbol{e}_i) = \Gamma^T(\tau, \boldsymbol{X})\Gamma(\tau, \boldsymbol{X}), \qquad i = 1, 2$$

can be found from Eq. (17):

$$\Theta(\tau, \boldsymbol{e}_1) = \frac{g(\alpha)}{\sqrt{2}}\begin{bmatrix} h(\alpha)+\cos\alpha & -h(\alpha)+\cos\alpha \\ h(\alpha)-\cos\alpha & h(\alpha)+\cos\alpha \end{bmatrix}$$

$$\Theta(\tau, \boldsymbol{e}_2) = g(\alpha)\begin{bmatrix} h(\alpha) & \cos\alpha \\ -\cos\alpha & h(\alpha) \end{bmatrix}$$

To check Eq. (18), we compute the following matrices:

$$\Theta(\tau, \boldsymbol{e}_1)\Lambda_{,1}\Theta^T(\tau, \boldsymbol{e}_1)\cos\alpha$$

$$= -\frac{g^2(\alpha)\cos\alpha}{2}\begin{Bmatrix} [h(\alpha)+\cos\alpha]^2 & [h^2(\alpha)-\cos^2\alpha] \\ [h^2(\alpha)-\cos^2\alpha] & [h(\alpha)-\cos\alpha]^2 \end{Bmatrix}$$

$$\Theta(\tau, \boldsymbol{e}_2)\Lambda_{,2}\Theta^T(\tau, \boldsymbol{e}_2)\sin\alpha$$

$$= g^2(\alpha)\sin\alpha\begin{bmatrix} h^2(\alpha)+3\cos^2\alpha & 2h(\alpha)\cos\alpha \\ 2h(\alpha)\cos\alpha & 3h^2(\alpha)+\cos^2\alpha \end{bmatrix}$$

By direct computation or use of Mathematics software, we can get

$$\Theta(\tau, \boldsymbol{e}_1)\Lambda_{,1}\Theta^T(\tau, \boldsymbol{e}_1)\cos\alpha + \Theta(\tau, \boldsymbol{e}_2)\Lambda_{,2}\Theta^T(\tau, \boldsymbol{e}_2)\sin\alpha$$

$$= \begin{bmatrix} \lambda^{(1)}_{,\tau} & 0 \\ 0 & \lambda^{(2)}_{,\tau} \end{bmatrix} = \Lambda_{,\tau}$$

but Eq. (31) does not hold; consequently, we can conclude that

$$\boldsymbol{Z}_{,\tau} \neq \boldsymbol{Z}_{,1}\Theta^T(\tau, \boldsymbol{e}_1)\cos\alpha + \boldsymbol{Z}_{,2}\Theta^T(\tau, \boldsymbol{e}_2)\sin\alpha$$

To use the inverse sensitivity method, we must employ Eqs. (43–45); in the present case it is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} \Delta p \\ \Delta q \end{Bmatrix} = \begin{Bmatrix} \bar{\lambda}^{(1)} - \lambda \\ \bar{\lambda}^{(2)} - \lambda \end{Bmatrix}$$

where

$$a_{11} = \lambda^{(1)}_{,1}\left[\theta^{11}(\tau, \boldsymbol{e}_1)\right]^2 + \lambda^{(2)}_{,1}\left[\theta^{12}(\tau, \boldsymbol{e}_1)\right]^2$$

$$a_{12} = \lambda^{(1)}_{,2}\left[\theta^{11}(\tau, \boldsymbol{e}_2)\right]^2 + \lambda^{(2)}_{,2}\left[\theta^{12}(\tau, \boldsymbol{e}_2)\right]^2$$

$$a_{21} = \lambda^{(1)}_{,1}\left[\theta^{21}(\tau, \boldsymbol{e}_1)\right]^2 + \lambda^{(2)}_{,1}\left[\theta^{22}(\tau, \boldsymbol{e}_1)\right]^2$$

$$a_{22} = \lambda^{(1)}_{,2}\left[\theta^{21}(\tau, \boldsymbol{e}_2)\right]^2 + \lambda^{(2)}_{,2}\left[\theta^{22}(\tau, \boldsymbol{e}_2)\right]^2$$

For checking the effectiveness of the present method, we assume the target values of the parameters to be

$$p_t = p_0 + 0.03, \qquad q_t = q_0 - 0.04$$

The eigenvalues of target system $[\boldsymbol{K}(\boldsymbol{p}_t), \boldsymbol{M}(\boldsymbol{p}_t)]$ are

$$\bar{\lambda}^{(1)} = 0.8691, \qquad \bar{\lambda}^{(2)} = 0.9488$$

and the directional cosines are

$$\cos\alpha = \tfrac{3}{5}, \qquad \sin\alpha = -\tfrac{4}{5}$$

Substituting the data into the preceding equations, we can get

$$\begin{bmatrix} -0.6756 & 2.9363 \\ -0.3244 & 1.0637 \end{bmatrix}\begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = \begin{bmatrix} -0.1309 \\ -0.0512 \end{bmatrix}$$

The solution of this equation is found to be

$$\Delta\boldsymbol{p}^T = [\Delta p \quad \Delta q] = [0.0467 \quad -0.0339]$$

If the off-diagonal entries of $\Lambda_{,\tau}$ in Eq. (18) were also considered in the solution of Eq. (43), the result for $\Delta\boldsymbol{p}$ is very similar. Thus the corrected point is

$$\boldsymbol{p}_c = \boldsymbol{p}_0 + \Delta\boldsymbol{p}$$

The distance to target point has reduced about 66.7%.

Conclusion

An inverse sensitivity method has been developed for complementing and correcting Pesek's results.[5] Although we have confined our attention to two simple examples, the same procedure can be carried through for more complicated structures. The size and complexity of realistic models means that it is essential to use a digital computer to solve the resulting equations. Finally we point out that this method can be extended to treat the nondefective repeated eigenvalue having equal partial derivatives of first order using formulas presented by Refs. 4, 7, and 8.

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